The CAPM and Index Models

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Financial Instruments & Markets
[Financial Investment and Corporate Finance]
Academic year 2022-2023

October 17, 2022

Outline of talk

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A Single-Index Stock Market

- We know the distinction between systematic and firm-specific risk;
- Systematic risk is macroeconomic, affecting all securities, while firm-specific risk factors affect only one particular firm or, at most, a cluster of firms;
- **Index models** are statistical models designed to estimate these two components of risk for a particular security or portfolio;
- The first to use and index model to explain the benefits of diversification was the Nobel Prize winner, **William F. Sharpe (1963)**;
- We will investigate his major work (the Capital Asset Pricing Model (CAPM)) in the next slides;

A Single-Index Stock Market

- The popularity of index models is due to their practicality;
- As an example: to construct the efficient frontier from a universe of 100 risky securities, we would need to estimate 100 expected returns, 100 variances, and 100*99/2=4,950 covariances;
- A universe of 1,000 securities would require estimates of 1,000*999/2=49 co-variances, as well as 1,000 expected returns and variances;
- Assuming that one common factor is responsible for all the co-variability of stock returns, with all other variability due to firm-specific factors, dramatically simplifies the analysis;

A Single-Index Stock Market

- Let us denote R_i the **excess return** on a security, that is, the rate of return in excess of the risk-free rate: $R_i = r_i - r_f$. Then we can express the distinction between macroeconomic and firm-specific factors by decomposing this excess return in some holding period into three components:

$$R_i = \beta_i R_M + \varepsilon_i + \alpha_i \tag{1}$$

- The first two terms on the right-hand side of Equation 1 reflect the impact of two sources of uncertainty. R_M is the excess return on a broad market index (i.e., S&P500 is commonly used for this purpose), so variation in this term reflects the influence of economy wide or macroeconomic events that generally affect all stocks to greater or lesser degrees. β_i is the **security' beta** and is typical response of that particular stock' excess return to changes in the market index' excess return;

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- Beta measures a stock' comparative sensitivity to macroeconomic news. If $\beta_i > 1$ indicates a stock with greater sensitivity to the economy than the average stock. These are known as **cyclical stocks**. If $\beta_i < 1$ indicates below average sensitivity and therefore are knows as **defensive stocks**;
- Recall that the risk attributable to the stock' exposure to uncertain market returns is called market or **systematic risk**, because it relates to the uncertainty that pervades the whole economic system;
- The term ϵ_i represents the impact of **firm-specific** or **residual risk**. The expected value of ϵ_i is zero, as the impact of unexpected events must average out to zero. Both residual risk and systematic risk contribute to the total volatility of returns;
- The term α_i is not a risk measure. Instead it represents the expected return on the stock beyond any return induced by movements in the market index. This term is called the security **alpha**;

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- A positive alpha is attractive to investors and suggests an underpriced security;
- Among securities with identical β_i to the market index, securities with higher alpha values will offer higher expected returns;
- Conversely, stocks with negative alphas are apparently overpriced; for any value of β_i, they offer lower expected returns;
- To summarize, the index model separates the realized rate of return on a security into macro (systematic) and micro (firm-specific) components;

- The excess rate of return on each security is the sum of three components:

Table 1: Components of the excess rate of return.

	Symbol
1. The component of return due to movements in the overall market (as represented by the index R_M); β_i is the security' responsiveness to the market	β_i;R_M
2. The component attributable to unexpected events that are relevant only to this security (firm-specific).	ε_i
3. The stock' expected excess return if the market factor is neutral, that is, if the market-index excess return is zero.	α_i

- Because the firm-specific component of the stock return in uncorrelated with the market return, we can write the variance of the excess return of the stock as:

A Single-Index Stock Market

$$\begin{aligned}\sigma^2(R_i) =& \sigma^2(\alpha_i + \beta_i R_M + \varepsilon_i) \\ =& \sigma^2(\beta_i R_M) + \sigma^2(\varepsilon_i) \\ =& \beta_i^2 \sigma^2(R_M) + \sigma^2(\varepsilon_i) \\ =& Systematic\ risk + Firm - specific\ risk\end{aligned} \tag{2}$$

- Therefore, the total variance of the rate of return of each security is a sum of two components:
  1. the variance attributable to the uncertainty of the entire market. This variance depends on both the variance of and the β of the stock on R_M;
  2. the variance of the firm-specific return, ϵ_i, which is independent of market performance.
- The single-index model is convenient. It relates security returns to a market index that investors follow. Moreover, as we soon shall see, its usefulness goes beyond mere convenience.

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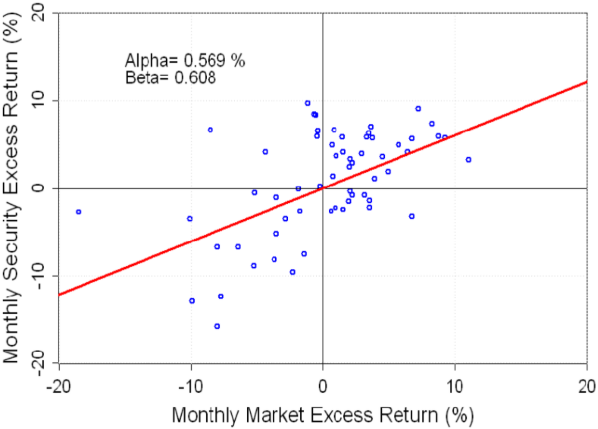
- The Equation 1, $R_i = \alpha_i + \beta_i R_M + \varepsilon_i$, may be interpreted a single-variable regression equation of R_i on the market excess return R_M;
- The R_i, which is the excess return on the security, is the dependent variable that is to be explained by the regression;
- α_i is the intercept; β_i is the regression (slope) coefficient, multiplying the independent (explanatory) variable R_M; ϵ_i is the residual (unexplained) return;
- We plot this regression in Figure 1, which shows a scatter diagram for the excess return of a security i against the excess return of the market index;

Figure 1: Scatter diagram for security *i*

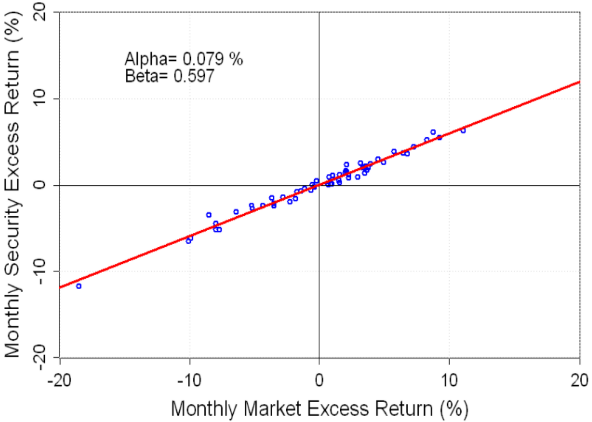


- The horizontal axis of the scatter diagram measures the explanatory variable, here the market excess return, R_M;
- The vertical axis measures the dependent variable, here the (monthly) excess return, R_i, of security i;
- Regression analysis uses a sample of historical returns to estimate the coefficients, α and β, of the index model;
- The analysis finds the regression line, shown in Figure 1, that minimizes the sum of the squared deviations around it;
- We say the regression line "bets fits" the data in the scatter diagram;
- The line is called the **Security Characteristic Line**, or SCL;
- In this case, the beta is equal to 0.608, which means that when the market rises or falls by 1% the security i tends to rise or fall by about 0.608%.

- The regression relationship between the security i return and the market return is far from perfect;
- Many of the scatter plot points showing the actual returns for i and the market portfolio fall far from the best fit line;
- According to the CAPM these "errors" should average out in a diversified portfolio;
- Let's have a look at the returns of fund i. The fund is composed of a diverse portfolio of both stocks and bonds, and the beta of the fund is very similar to the beta of the previous security;
- We plot this regression in Figure 2, which shows a scatter diagram for the excess return of a security fund i against the excess return of the market index;

Statistical and Graphical Representation

Figure 2: Scatter diagram for fund *i*



- All the scatter plot points in Figure 2 fall very close to the regression line;
- We do not see the large errors that were seen in first graph;
- The "diversifiable" risk has been eliminated and we are left only with a "non-diversifiable" exposure to overall market risk which is defined by the average beta of the securities in the portfolio;
- Let's provide some examples;

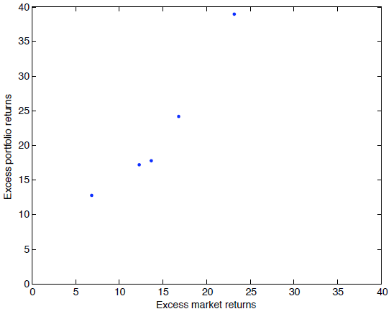
- We have the following data on a fund manager' portfolio excess returns and the excess return on the market index (e.g., the FTSE-100);

Table 2: Components of the excess rate of return.

Year	Portfolio excess returns (y)	Market excess returns (x)
1	17.8	13.7
2	39	23.2
3	12.8	6.9
4	24.2	16.8
5	17.2	12.3

- In order to get a first clue as to the nature of the relationship between these two variables, let us build a scatterplot of the data.

Figure 3: Scatter plot data of Table 2



- There is a positive relationship here. When excess market returns are high (low), excess returns on this portfolio tend to be high (low);

- Begin with a generic equation for a straight line relationship between y and x:

$$y = \alpha + \beta x \tag{3}$$

- However, this line is completely deterministic. Is this realistic? No! So we add a random error term (ε) to the RHS of the equation;

$$y_i = \alpha_i + \beta x_i + \varepsilon_i \tag{4}$$

where $i = 1, 2, 3, ..., N$ and N is the number of observations.

- **Why include the error term?** To capture: [i] factors that affect y but which are missing from our set of regressors x; [ii] error in the measurement of the y variable; [iii] random and non-systematic influences that cannot be modeled;
- **So, how do we estimate the values for α and β? Least squares criterion**: we choose estimates for α and β such that the sum of the squares of the distances between the data points and the line implied by the estimates of α and β is minimized;
- We define:
  1. a to be the estimate of α;
  2. b to be the estimate of β;
  3. $\widehat{y}_i$ to be the estimated value of observation i from the regression line:

$$\widehat{y}_i = a + bx_i \tag{5}$$

  4. $\widehat{\epsilon}_i$ to be the residual associated with observation i, equal to $(y_i - \widehat{y}_i)$.

- Here's our regression model again: $y_i = \alpha_i + \beta x_i + \varepsilon_i$, and after substituting: $R_i = \alpha_i + \beta_i R_M + \varepsilon_i$;
- Assume that we have used least squares to find estimates of α_i and β_i. As above, we call these a and b;
- **What do a and b tell us?** Well, they allow us to build a fitted regression line: $\widehat{y}_i = a + bx_i$;
- Now we can see that a and b are the coefficients that determine the location of the estimated straight line relationship between fitted values for y and x. Thus:
    1. a is the estimated intercept. It tells us the value that y is predicted to take if x takes the value zero;
    2. b is the estimated slope. It tells us that if x increases (decreases) by 1 unit, then we would expect y to increase (decrease) by b units.

- **Scenario:** take our previous example where we were given a fund' excess returns and excess returns on the market;
- **First question:** what is the relationship between the returns on the fund and the return on the market?
- **Answer:** applying least squares, we get: a=-1.74; b=1.64, where y_i is the return on the fund and x_i is the return on the market;
- These numbers imply that: $\widehat{y_i} = -1.74 + 1.64x_i$;
- **Interpretation:** what do our numbers mean:
  1. **Intercept:** when the return on the market (x_i) is zero, we expect the fund to return -1.74%;
  2. **Slope:** for every one percentage point increase in the market return, we expect the fund to return an extra 1.64%. Similarly, for every 1% fall in the market, we expect the fund return to drop by 1.64%.

- **Question:** if an analyst tells you that next year (s)he expects the market to return 20% above the risk-free rate, what is your estimate of the return on the fund?;
- **Solution:**
  1. Start with the fitted regression line derived above: $\widehat{y}_i = -1.74 + 1.64x_i$;
  2. Then, as our analyst has told you to expect x_i=20, our estimate of the fund return is: $\widehat{y}_i = -1.74 + 1.64 * 20 = 31.06\%$;
  3. Now, if your analyst changes his forecast, you can go through the same process again to get a new estimate of the fund's return next year...

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- **Now, is your turn.**
- We start with the monthly series of:
  1. Amazon, Apple, Microsoft and Wells Fargo & Co stock prices;
  2. S&P Index;
  3. T-bills rate;
  4. Time period: 1999:12-2022:9.
- Please, generate monthly excess returns on Amazon, Apple, Microsoft and Wells Fargo' stock and the market index;
- The period 1999:12-2022:9 includes some sub-periods which could become interesting for your empirical analysis;
- Please, compute a descriptive statistics of your data;
- Please, estimate the values for α and β via the Least squares criterion, for both the whole time period and for each sub-periods identified;
- Provide a preliminary interpretation of the empirical results.