THE BINOMIAL OPTION PRICING MODEL

LET'S START BY COMPUTING THE PRICE OF A SINGLE PERIOD OPTION IN A VERY SIMPLE WORLD.

WE WILL VALUE THE OPTION BY FIRST CONSTRUCTING A REPLICATING PORTFOLIO, A PORTFOLIO OF OTHER SECURITIES THAT HAS THE SAME (EXACTLY) VALUE IN ONE PERIOD AS THE OPTION.

THEN, BECAUSE THEY HAVE THE SAME PAYOFFS, THE LAW OF ONE PRICE IMPLIES THAT THE CURRENT VALUE OF THE CALL AND THE REPLICATING PORTFOLIO MUST BE EQUAL.

EX. CONSIDER A EUROPEAN CALL OPTION THAT EXPIRES IN ONE PERIOD AND HAS AN EXERCISE PRICE OF 50. ALSO ASSUME THAT THE STOCK PRICE TODAY IS EQUAL TO 50. (ASSUME THAT THE STOCK PAYS NO DIVIDEND).
IN ONE PERIOD, THE STOCK PRICE WILL EITHER RISE BY 10 OR FALL BY 10.
THE ONE-PERIOD RISK-FREE RATE IS 6%.

WE CAN SUMMARIZE THIS INFORMATION ON A $\boxed{\text{BINOMIAL TREE}}$, WHICH IS A TIMELINE WITH TWO BRANCHES AT EVERY DATE REPRESENTING THE POSSIBLE EVENTS THAT COULD HAPPEN AT THOSE TIMES:

0		1		
		STOCK	BOND	CALL
STOCK 50		60	1.06	$\max(60-50,0)=10$
BOND 1		40	1.06	$\max(40-50,0)=0$

THE BINOMIAL TREE CONTAINS ALL THE INFORMATION WE KNOW:

- THE VALUE OF THE STOCK
- THE VALUE OF THE BOND
- THE VALUE OF CALL OPTIONS

$\Big\}$ FOR EACH STATE IN ONE PERIOD

REGARDING THE BOND, FOR SIMPLICITY, WE ASSUME THE BOND PRICE TODAY IS 1, SO IN ONE PERIOD IT WILL BE WORTH 1.06.

WE DEFINE THE STATE IN WHICH THE STOCK PRICE GOES UP (TO 60) AS THE UP, AND THE STATE IN WHICH THE STOCK PRICE GOES DOWN (TO 40) AS THE DOWN STATE.

IN ORDER TO COMPUTE THE VALUE OF THE OPTION USING THE LAW OF ONE PRICE, WE MUST SHOW THAT WE CAN REPLICATE ITS PAYOFFS USING A PORTFOLIO OF THE STOCK AND THE BOND.

TO THIS END, LET Δ BE THE NUMBER OF SHARES OF STOCK WE PURCHASE, AND LET B BE OUR INITIAL INVESTMENT IN BONDS.

TO ~~[illegible]~~ CREATE A CALL OPTION USING THE STOCK AND THE BOND, THE VALUE OF THE PORTFOLIO CONSISTING OF THE STOCK AND BOND MUST MATCH THE VALUE OF THE OPTION IN EVERY POSSIBLE STATE.

THUS, IN THE UP STATE, THE VALUE OF THE PORTFOLIO MUST BE 10 (THE VALUE OF THE CALL IN THAT STATE):

$$60\Delta + 1.06B = 10 \qquad (1)$$

N THE DOWN STATE, THE VALUE OF THE PORTFOLIO MUST BE ZERO (THE VALUE OF THE CALL IN THAT STATE):

$$40\Delta + 1.06B = 0 \qquad (2)$$

EQUATIONS (1) AND (2) ARE TWO SIMULTANEOUS EQUATIONS WITH TWO UNKNOWNS, Δ AND B. THE SOLUTION IS

$$\Delta = 0.5$$
$$B = -18.8679$$

A PORTFOLIO WITH 0.5 SHARE OF STOCK AND APPROXIMATELY 18.87 WORTH OF BONDS (i.e., WE HAVE BORROWED 18.87 AT A 6% INTEREST RATE) WILL HAVE A VALUE IN ONE PERIOD THAT MATCHES THE VALUE OF THE CALL EXACTLY. LET'S VERIFY THIS EXPLICITY:

$$60 \cdot 0.5 - 1.06 \cdot 18.87 = 10$$

$$40 \cdot 0.5 - 1.06 \cdot 18.87 = 0$$

BY THE LAW OF ONE PRICE, THE PRICE OF THE CALL OPTION TODAY MUST EQUAL THE CURRENT MARKET VALUE OF THE REPLICATING PORTFOLIO.

THE VALUE OF THE PORTFOLIO TODAY IS THE VALUE OF 0.5 SHARES AT THE CURRENT PRICE OF 50, LESS THE AMOUNT BORROWED

$$50\Delta + B = 50(.05) - 18.87 = 6.13$$

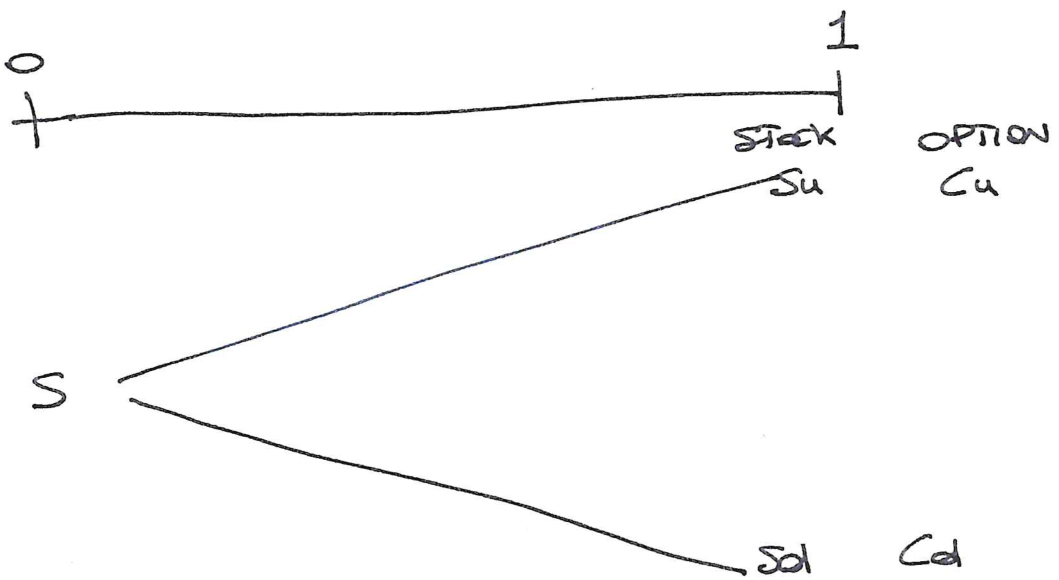
THUS, THE PRICE OF THE CALL TODAY IS 6.13.

THE BINOMIAL PRICING FORMULA

LET'S CONSIDER A MORE GENERAL EXAMPLE.

SUPPOSE THE CURRENT STOCK PRICE IS S, AND THE STOCK PRICE WILL EITHER GO UP TO S_u OR DOWN TO S_d NEXT PERIOD. THE RISK-FREE INTEREST RATE IS r_f.

LET'S DETERMINE THE PRICE OF AN OPTION THAT HAS A VALUE OF C_u IF THE STOCK GOES UP, AND C_d IF THE STOCK GOES DOWN:



FOR SIMPLICITY, WE DID NOT WRITE DOWN THE BOND PAYOFF SINCE IT EARNS A RETURN OF r_f IN EITHER CASE.

WHAT IS THE VALUE OF THE OPTION TODAY? ONCE AGAIN, WE MUST COMPUTE THE NUMBER OF SHARE Δ OF STOCK AND THE POSITION IN THE BOND, B, SUCH THAT THE PAYOFF OF THE REPLICATING PORTFOLIO MATCHES THE PAYOFF OF THE OPTION IF THE STOCK GOES UP OR DOWN:

$$S_u \Delta + (1+r_f)B = C_u \quad \text{and} \quad S_d \cdot \Delta + (1+r_f)B = C_d$$

SOLVING THESE TWO EQUATIONS FOR THE UNKNOWNS Δ AND B, WE GET THE GENERAL FORMULA FOR THE REPLICATING FORMULA IN THE BINOMIAL MODEL

REPLICATING PORTFOLIO IN THE BINOMIAL MODEL

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \quad \text{AND} \quad B = \frac{C_d - S_d \Delta}{1 + r_f}$$

CAN BE INTERPRETED AS THE SENSITIVITY OF THE OPTION'S VALUE TO CHANGE IN THE STOCK PRICE.

OPTION PRICE IN THE BINOMIAL MODEL

ONCE WE KNOW THE REPLICATING PORTFOLIO, WE CAN CALCULATE THE VALUE C OF THE OPTION TODAY, AS THE COST OF THIS PORTFOLIO:

$$C = S \cdot \Delta + B$$

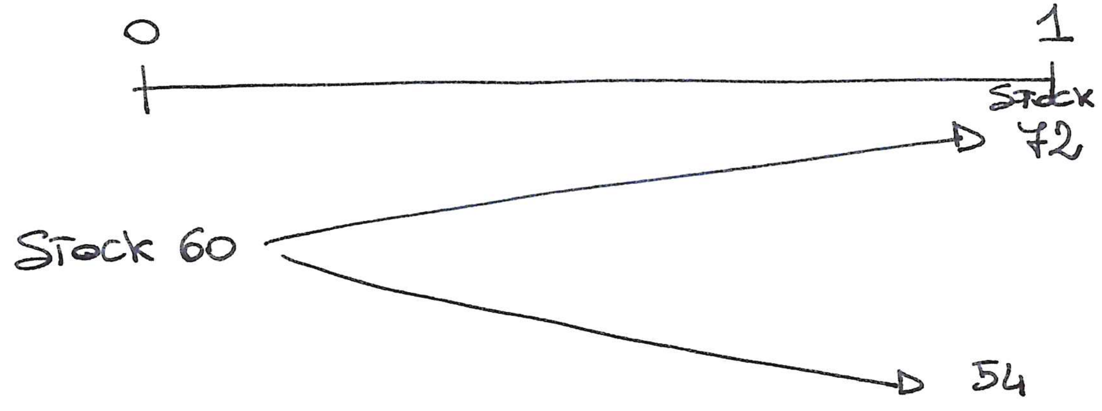
EQUATION FOR REPLICATING PORTFOLIO IN THE BINOMIAL MODEL, AND THE EQUATION FOR PRICING THE OPTION IN THE BINOMIAL MODEL ARE QUITE POWERFUL.

THEY DO NOT REQUIRE THAT THE OPTION WE ARE VALUING IS A CALL OPTION. WE CAN USE THEM TO VALUE ANY SECURITY WHOSE PAYOFF DEPENDS ON THE STOCK PRICE.

VALUING A PUT OPTION

SUPPOSE A STOCK IS CURRENTLY TRADING FOR 60, AND IN ONE PERIOD WILL EITHER GO UP BY 20% OR FALL BY 10%. . IF THE ONE PERIOD Rf IS 3%, WHAT IS THE PRICE OF A EUROPEAN PUT OPTION THAT EXPIRES IN ONE PERIOD AND HAS AN EXERCISE PRICE OF 60?

OUR BINOMIAL TREE:



PUT

$\max(60-72,0)=0$

$\max(60-54,0)=6$

WE CAN SOLVE FOR THE VALUE OF THE PUT BY USING $Cu=0$ and $Cd=6$

THEREFORE:

$$\Delta = \frac{Cu-Cd}{Su-Sd} = \frac{0-6}{72-54} = -0.3333 \quad \text{and} \quad B = \frac{Cd - Sd \cdot \Delta}{1+r_f} = \frac{6-54(-0.3333)}{1.03}$$

$B = 23.30$

THIS PORTFOLIO IS COMPOSED OF 0.3333 SHARES OF THE STOCK (ALTERNATIVELY, THIS PORTFOLIO IS SHORT 0.3333 SHARES OF THE STOCK), AND HAS 23.30 INVESTED IN THE Rf BOND.

LET'S CHECK THAT IT REPLICATES THE PUT IF THE STOCK GOES UP OR DOWN:

$$72(-0.3333) + 1.03(23.30) = 0 \quad \text{AND} \quad 54(-0.3333) + 1.03(23.30) = 6$$

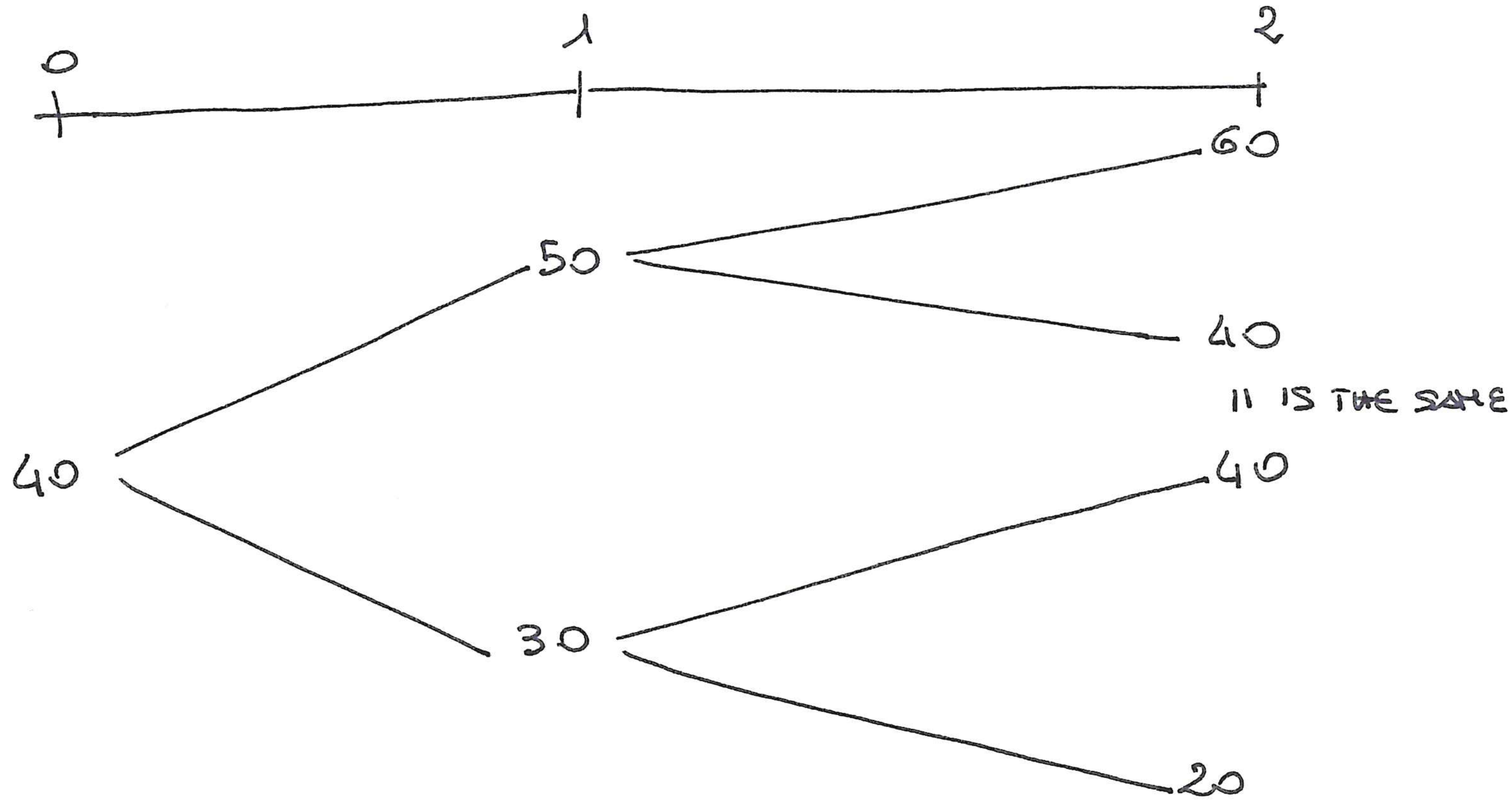
THUS THE VALUE OF THE PUT IS THE INITIAL COST OF THIS PORTFOLIO:

$$\text{PUT VALUE} = C = S \cdot \Delta + B$$

$$= 60(-0.3333) + 23.30 = 3.30$$

A MULTIPERIOD MODEL

LET'S CONSIDER A TWO-PERIOD BINOMIAL TREE (BINOMIAL LATTICE) FOR THE STOCK PRICE:

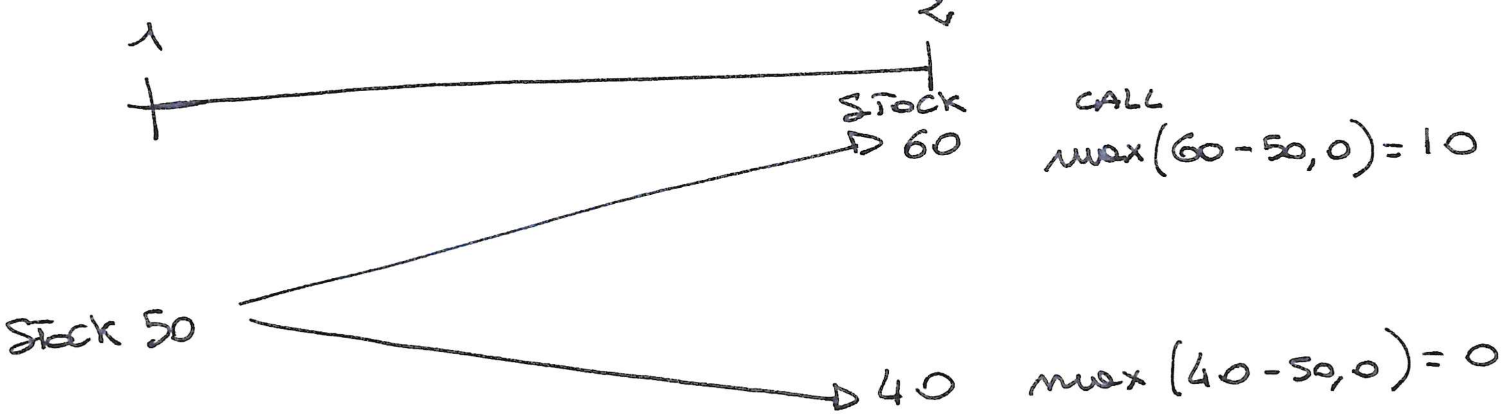


THE KEY PROPERTY OF THE BINOMIAL MODEL IS THAT IN EACH PERIOD, THERE ARE ONLY TWO POSSIBLE OUTCOMES (THE STOCK EITHER GOES UP OR DOWN). BY ADDING AN ADDITIONAL PERIOD, THE NUMBER OF POSSIBLE STOCK PRICES GOES UP OR DOWN, i.e. AT THE END THE NUMBER OF POSSIBLE STOCK PRICES HAS INCREASED

LET'S ASSUME THAT $r_f = 6\%$ PER PERIOD AND CONSIDER HOW TO PRICE A CALL OPTION WITH A STRIKE PRICE OF 50 THAT EXPIRES IN TWO PERIODS.

TO COMPUTE THE VALUE OF AN OPTION IN A MULTIPERIOD MODEL, WE START AT THE END OF THE TREE AND WORK BACKWARD.

AT THE TIME 2, THE OPTION EXPIRES, SO ITS VALUE IS EQUAL TO ITS INTRINSIC VALUE. IN THIS CASE, THE CALL WILL BE WORTH 10 IF THE STOCK PRICE GOES UP TO 60, AND WILL BE WORTH ZERO OTHERWISE.



WHAT IS THE VALUE OF THE OPTION IF THE STOCK PRICE HAS GONE UP TO 50 AT TIME 1?

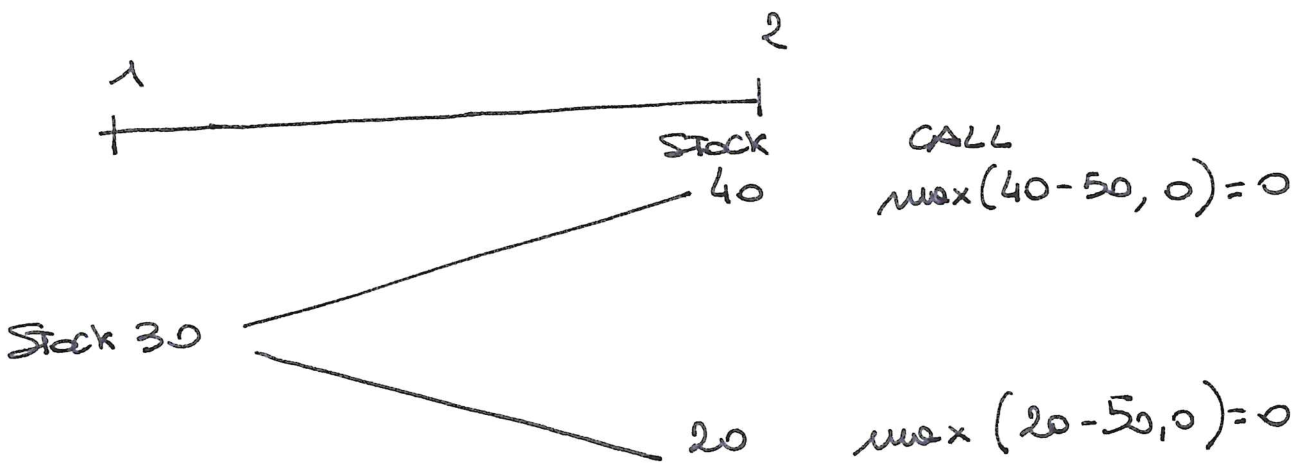
WE COMPUTE THE REPLICATING PORTFOLIO AS HAVING $\Delta = 0.5$ AND $B = -18.87$

$$\Delta = \frac{10 - 0}{60 - 40} = 0.5 \quad \text{AND} \quad B = \frac{0 - 40(0.5)}{1.06} = -18.86$$

AT TIME 1, THE VALUE OF THE CALL OPTION IS:

$$C = S\Delta + B = 50(0.5) - 18.86 = 6.13$$

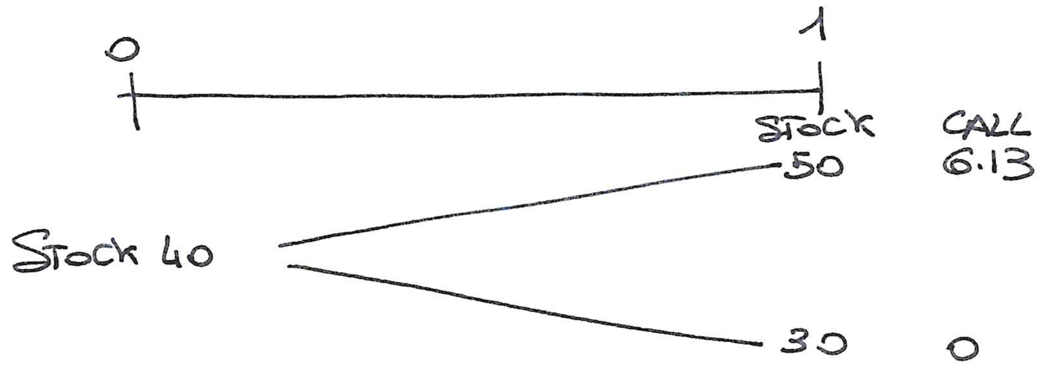
WHAT IF THE STOCK PRICE HAS DROPPED TO 30 AT TIME 1? IN THAT CASE, THE BINOMIAL TREE FOR THE NEXT PERIOD IS:



THE OPTION IS WORTHLESS IN BOTH STATES AT TIME 2, SO THE VALUE OF THE OPTION IN THE DOWN STATE AT TIME 1 MUST ALSO BE ZERO. THUS REPLICATING THE PORTFOLIO IS SIMPLY $\Delta = 0$ AND $B = 0$

GIVEN THE VALUE OF THE CALL OPTION IN EITHER STATE AT TIME 1, WE CAN NOW WORK BACKWARD AND COMPUTE THE VALUE OF THE CALL AT TIME 0.

IN THAT CASE, WE CAN WRITE THE BINOMIAL TREE OVER THE NEXT PERIOD AS FOLLOWS:



IN THIS CASE, THE OPTION CALL VALUS AT THE END OF THE TREE (TIME 1) ARE NOT THE FINAL PAYOFFS OF THE OPTION, BUT ARE THE VALUES OF THE OPTION ONE PERIOD PRIOR TO EXPIRATION.

NONETHELESS, WE CAN USE THE SAME BINOMIAL FORMULAS TO CALCULATE THE REPLICATING PORTFOLIO AT TIME 0, WHICH IS A PORTFOLIO WHOSE VALUE WILL MATCH THE VALUE OF THE OPTION AT TIME 1.

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{6.13 - 0}{50 - 30} = 0.3065$$

$$B = \frac{C_d - S_d \cdot \Delta}{1 + r_f} = \frac{0 - 30(0.3065)}{1.06} = -8.67$$

THE INITIAL VALUE OF THE CALL OPTION IS EQUAL TO THE INITIAL COST OF THIS PORTFOLIO:

$$C = S \cdot \Delta + B = 40(0.3065) - 8.67 = 3.59$$

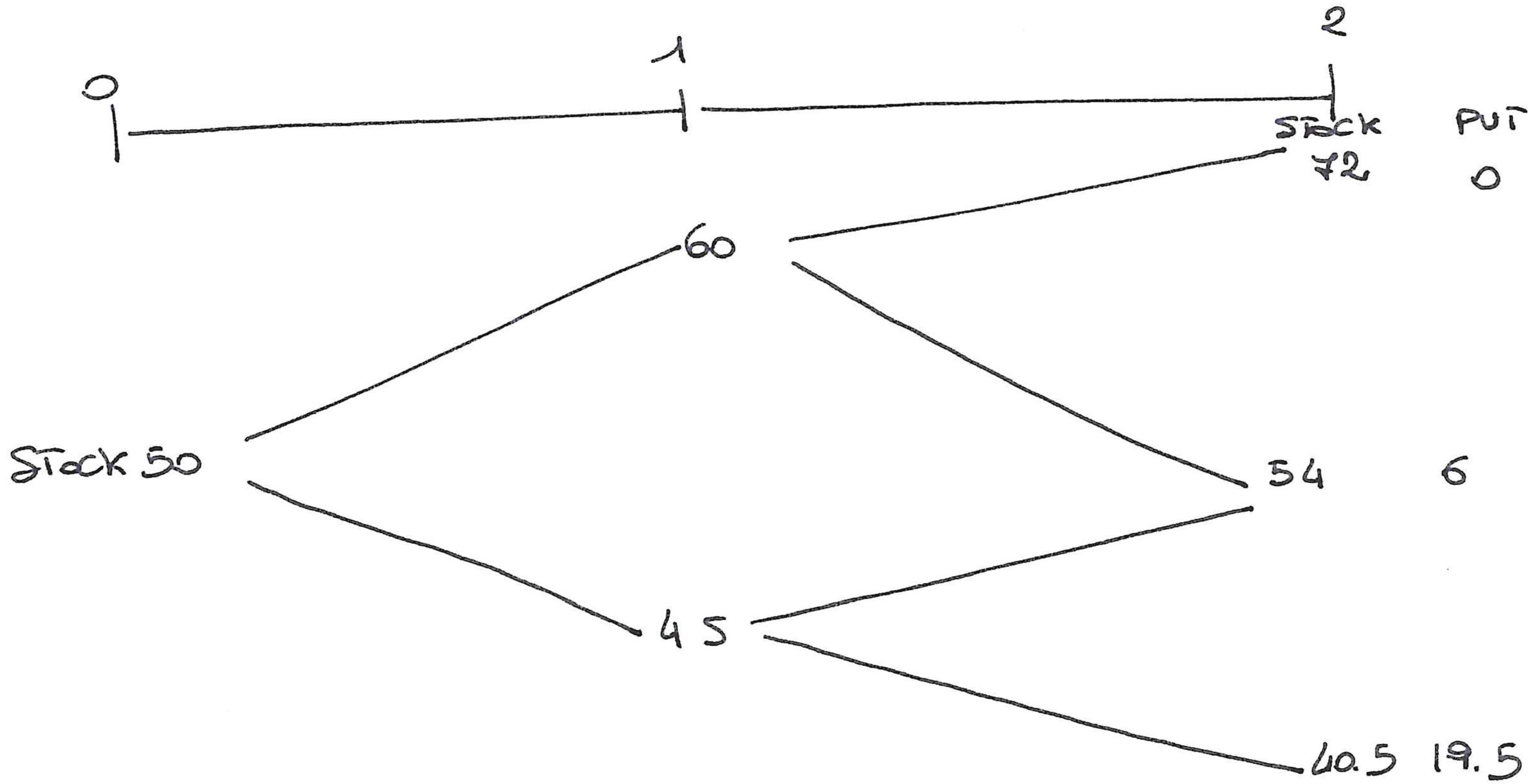
THEREFORE, THE INITIAL VALUE OF THE CALL OPTION AT TIME 0 IS 3.59.

THE IDEA THAT YOU CAN REPLICATE THE OPTION PAYOFF BY DYNAMICALLY TRADING IN A PORTFOLIO OF THE UNDERLYING STOCK AND A RISK-FREE BOND WAS ONE OF THE MOST IMPORTANT CONTRIBUTIONS OF THE ORIGINAL BLACK-SCHOLES PAPER. TODAY, THIS KIND OF REPLICATION STRATEGY IS CALLED A DYNAMIC TRADING STRATEGY.

EXAMPLE

SUPPOSE THAT THE CURRENT PRICE OF MILOS STOCK IS 50 PER SHARE. IN EACH OF THE NEXT TWO YEARS, THE STOCK PRICE WILL EITHER INCREASE BY 20% OR DECREASE BY 10%. . $r_f = 3\%$ AND WILL REMAIN CONSTANT.
CALCULATE THE PRICE OF A TWO-YEAR EUROPEAN PUT OPTION ON MILOS STOCK WITH A STRIKE PRICE 60.

THE BINOMIAL TREE FOR THE STOCK PRICE, TOGETHER WITH THE FINAL PAYOFFS OF THE PUT OPTION, IS:



F THE STOCK GOES UP TO 60 AT TIME 1, WE ARE IN EXACTLY THE SAME SITUATION AS WE SAW IN THE PREVIOUS EXAMPLE. THE VALUE OF THE PUT OPTION S 3.30.

F THE STOCK GOES DOWN TO 45 AT TIME 1, AT TIME 2 THE PUT OPTION WILL BE WORTH EITHER 6 IF THE STOCK GOES UP OR 19.50 IF THE STOCK GOES DOWN.

THERE FORE ;

$$\Delta = \frac{Cu - Cd}{Su - Sd} = \frac{6 - 19.5}{54 - 40.5} = -1 \quad \text{AND} \quad B = \frac{Cd - Sd\,\Delta}{1 + rf} = \frac{19.5 - 40.5(-1)}{1.03} = 58.25$$

THIS PORTFOLIO IS SHORT ONE SHARE OF THE STOCK AND HAS 58.25 INVESTED IN THE RISK-FREE BOND.

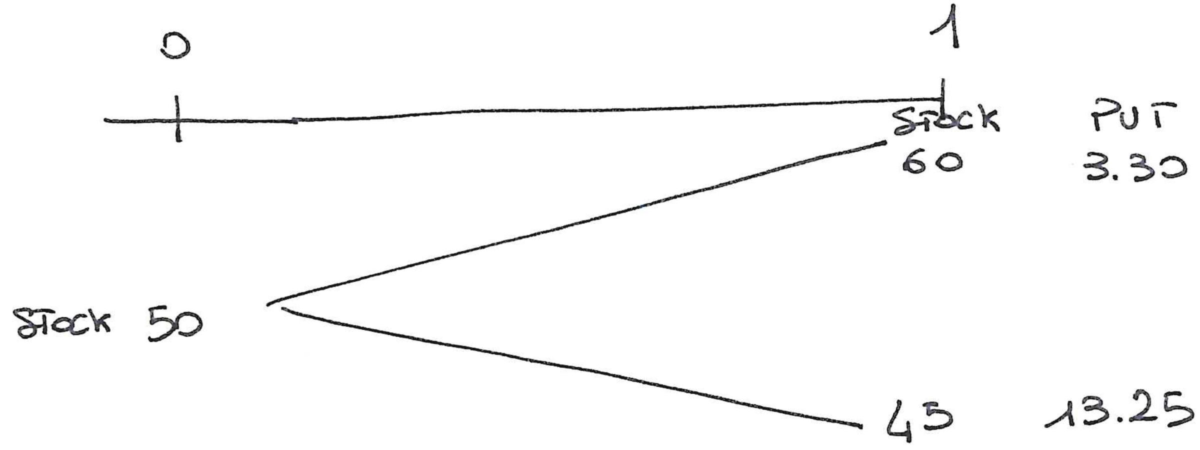
THE VALUE OF THE PUT IS:

$$\text{PUT VALUE} = C = S\Delta + B = 45(-1) + 58.25 = 13.25$$

NOW, WE CONSIDER THE VALUE OF THE PUT OPTION AT TIME 0.

IN PERIOD 1, WE HAVE CALCULATED THAT THE PUT WILL BE WORTH 3.30 IF THE STOCK GOES UP TO 60 AND 13.25 IF THE STOCK FALLS TO 45.

THE BINOMIAL TREE AT TIME 0 IS



THE REPLICATING PORTFOLIO AND PUT VALUE AT TIME 0 ARE

$$\Delta = \frac{Cu - Cd}{Su - Sd} = \frac{3.30 - 13.25}{60 - 45} = -0.6633$$

$$B = \frac{Cd - Sd \cdot \Delta}{1 + r_f} = \frac{13.25 - 45(-0.6633)}{1.03} = 41.84$$

$$\text{PUT VALUE} = C = S\Delta + B = 50(-0.6633) + 41.84 = 8.68$$

THUS, THE VALUE OF A EUROPEAN PUT OPTION AT TIME 0 IS 8.68

BLACK - SCHOLES OPTION PRICING MODEL

THE BLACK-SCHOLES OPTION PRICING MODEL CAN BE DERIVED FROM THE BINOMIAL OPTION PRICING MODEL BY MAKING THE LENGTH OF EACH PERIOD, AND THE MOVEMENT OF THE STOCK PRICE PER PERIOD, SHRINK TO ZERO AND LETTING THE NUMBER OF PERIODS GROW INFINITELY LARGE.

BEFORE STATING THE BLACK-SCHOLES FORMULA FOR THE PRICE OF AN OPTION, IT IS NECESSARY TO INTRODUCE SOME TERMINOLOGY. LET:

S = THE CURRENT PRICE OF THE STOCK;

T = THE NUMBER OF YEARS LEFT TO EXPIRATION;

K = THE EXERCISE PRICE;

σ = THE ANNUAL VOLATILITY (STD DEVIATION) OF THE STOCK'S RETURN.

THEN, THE VALUE, AT TIME t, OF A CALL OPTION ON A STOCK THAT DOES NOT PAY DIVIDENDS PRIOR TO THE OPTION'S EXPIRATION DATE IS GIVEN BY

$$\boxed{C = S \cdot N(d_1) - PV(K) \cdot N(d_2)}$$

THE VALUE, AT TIME t, OF A PUT OPTION ON A STOCK THAT DOES NOT PAY DIVIDENDS PRIOR TO THE OPTION'S EXPIRATION DATE IS GIVE BY:

$$\boxed{P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)]}$$

WHERE $PV(K)$ IS THE PRESENT VALUE (PRICE) OF A RISK-FREE ZERO-COUPON BOND THAT PAYS K ON THE EXPIRATION DATE OF THE OPTION, $N(d)$ IS THE CUMULATIVE NORMAL DISTRIBUTION, I.E. THE PROBABILITY THAT A NORMALLY DISTRIBUTED VARIABLE IS LESS THAN d;

$$d_1 = \frac{\ln[S/PV(K)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$N(d)$ HAS A MINIMUM VALUE OF 0 A MAXIMUM VALUE OF 1. IT CAN BE CALCULATED IN EXCEL BY USING THE FUNCTION "NORMDIST(d)".

VALUING A CALL OPTION WITH THE BLACK-SCHOLES

COMPUTE THE PRICE OF A 5.03 € CALL OPTION OF MILOS CORPORATION. THE VOLATILITY OF MILOS IS 65% PER YEAR, THE $r_f = 1\%$ PER YEAR, THE NUMBER OF DAYS LEFT TO EXPIRATION IS 148. THE CALL OPTION HAS A STRIKE PRICE OF 6 €

$$C = S \cdot N(d_1) - PV(K) \cdot N(d_2)$$

$$d_1 = \frac{\ln[S/PV(K)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\ln[5.03/PV(K)]}{0.65\sqrt{\frac{148}{365}}} + \frac{0.65\sqrt{\frac{148}{365}}}{2} =$$

$$PV(K) = \frac{6.00}{(1.01)^{\frac{148}{365}}} = 5.976 \qquad = \frac{\ln[5.03/5.976]}{0.65\sqrt{\frac{148}{365}}} + \frac{0.65\sqrt{\frac{148}{365}}}{2} = -0.209$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.209 - 0.65\sqrt{\frac{148}{365}} = -0.623$$

$$C = S \cdot \underbrace{N(d_1)} - PV(K) \cdot \underbrace{N(d_2)}$$

$$C = 5.03 \cdot 0.417 - 5.976 \cdot 0.267 = 0.5$$

VALUING A PUT OPTION WITH BLACK-SCHOLES FORMULA

COMPUTE THE PRICE OF A 5.03€ PUT OPTION OF MILOS CORPORATION. σ MILOS = 65% per year, r_f = 1% per year, The number of DAYS LEFT TO EXPIRATION IS 176. THE STRIKE PRICE IS 5€.

$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)]$$

$$PV(K) = \frac{5}{1.01^{\frac{176}{365}}} = 4.976$$

$$d_1 = \frac{\ln[S/PV(K)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\ln[5/4.976]}{0.65 \cdot \sqrt{\frac{176}{365}}} + \frac{0.65\sqrt{\frac{176}{365}}}{2} =$$

$$d_1 = 0.250$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.250 - 0.65\sqrt{\frac{176}{365}} = -0.201$$

$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)]$$

$$P = 4.976[1 - 0.420] - 5.03[1 - 0.599]$$

$$P = 0.87$$

OPTIONS AND CORPORATE FINANCE

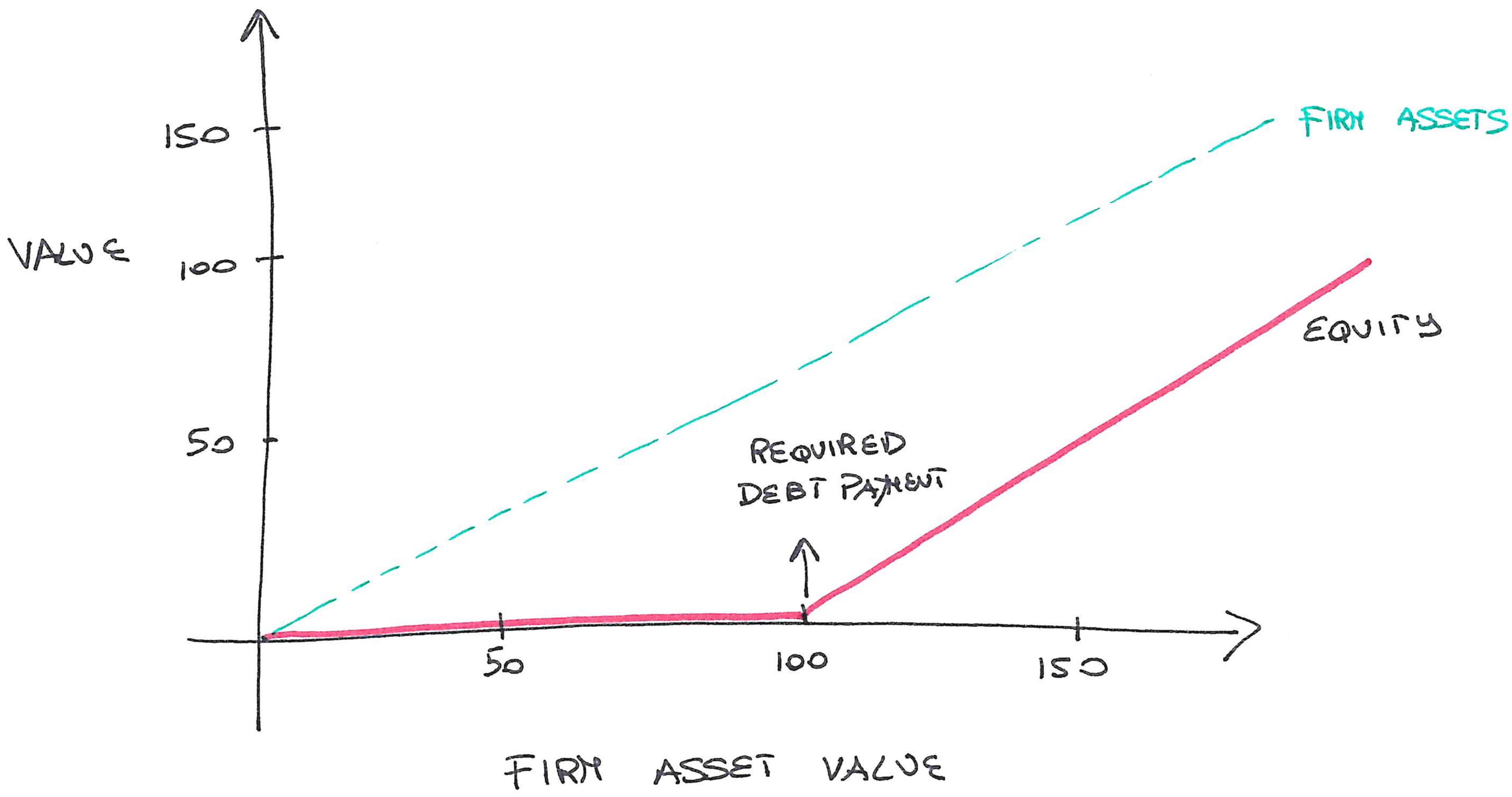
LET'S THINK OF A SHARE OF STOCK AS A CALL OPTION ON THE ASSETS OF THE FIRM WITH A STRIKE PRICE EQUAL TO THE VALUE OF DEBT OUTSTANDING.

TO ILLUSTRATE, CONSIDER A SINGLE-PERIOD WORLD IN WHICH AT THE END OF THE PERIOD THE FIRM IS LIQUIDATED.

IF THE FIRM'S VALUE DOES NOT EXCEED THE VALUE OF DEBT OUTSTANDING AT THE END OF THE PERIOD, THE FIRM MUST DECLARE BANKRUPTCY AND THE EQUITY HOLDERS RECEIVE NOTHING.

CONVERSELY, IF THE VALUE EXCEEDS THE VALUE OF DEBT OUTSTANDING, THE EQUITY HOLDERS GET WHATEVER IS LEFT ONCE THE DEBT HAS BEEN REPAID.

NOTICE THAT THE PAYOFF TO EQUITY LOOKS EXACTLY THE SAME AS THE PAYOFF OF A CALL OPTION.


VALUE
150
100
50
50
100
150
FIRM ASSETS
EQUITY
REQUIRED
DEBT PAYMENT
FIRM ASSET VALUE