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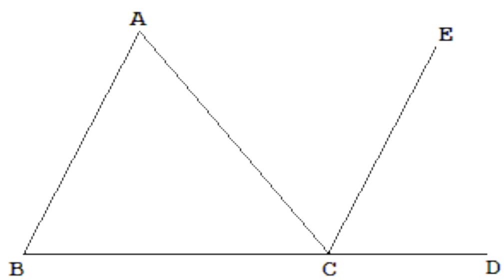
Bergamo/Pavia Masters seminars
May 29 - June 7 2024

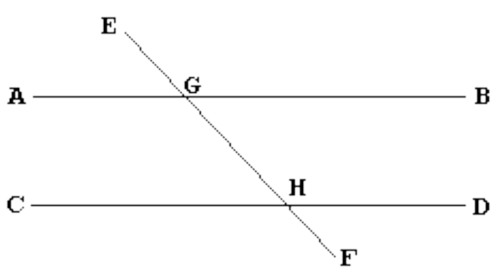
"Epistemology and Mathematics", Module 2 00
The Basic A Priori — Logic and Arithmetic

Handout 5 — The basic a priori and the role of imagination; Non-Cognitivism and Wittgenstein
Pre-reading: Wittgenstein, *Remarks on the Foundations of Mathematics*, part I, §§ 25-32

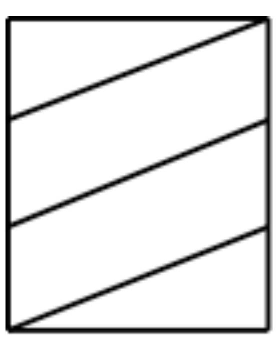
Here are some 'picture proofs'



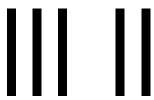
ABC + BCA + CAB = 180˚



Complementary angles are equal

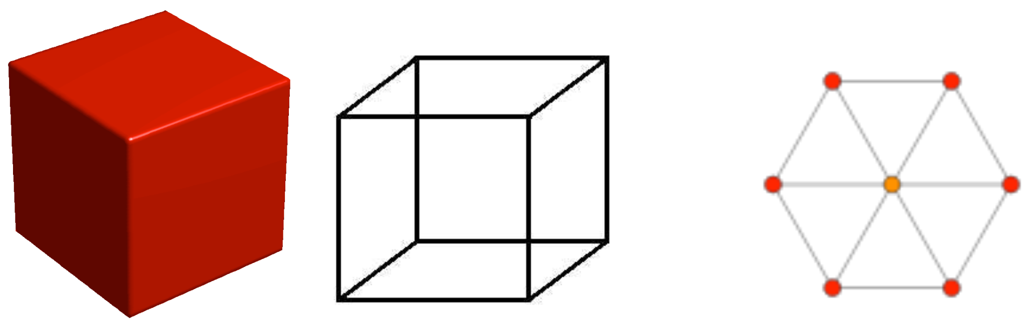


A rectangle can be composed of two similar triangles and two similar parallelograms



3+2=5

5×4 = 10×2



A cube has 12 edges

Suppose you are asked, How many edges has a cube?, and happen never before to have thought about it. You might happen to know an abstract definition of a cube: say, a three-dimensional solid bounded by six square faces, with three meeting at each vertex, and you might be able to reason deductively directly from that. But more likely you will imagine a cube, maybe like the red solid above, and perhaps reflect: "Well, I can see nine edges, and the six at the edge of the image are shared with the other side, so there will be three that I cannot see; hence twelve in all." Or you might reason that the edges of any two opposite square faces will total eight in number and will be connected one-to-one at the vertices by four more edges; and that will be all the edges — so twelve in all. Or you might think, "Well, there are six faces, and each has four edges. But each edge connects two faces—is used twice, as it were. So half four times six — twelve!" Or you might project the cube, Necker cube –style, as a 2D hexagon with interior connections like the right-most figure above and again proceed simply to count the lines connecting the nodes. And no doubt there are other ways of going about it. But what is striking about these and—one would expect— *any* way of going about it that does not amount merely to abstract deduction from an adequate formal definition is that *generalisations* feature in the reasoning—for instance: "Any cube presented to view in such a way that three of its faces are visible will have three unseen edges"; or "Any pair of opposite faces of a cube are connected by four edges"; or "Any cube has six faces and each of its edges connects two of them"; or "All the faces and edges of any cube can be projected into a two-dimensional figure, thus . . .". And these generalisations need not have been given as any explicit part of an original characterisation of the type, but nor need they be reasoned to as propositional entailments of some anterior characterisation. Indeed, the type may have been conveyed simply by pictures and illustrations. In that case there will be nothing to reason deductively from.

The notion of a *type* incorporates a distinction between essential and inessential features of its tokens; that distinction will ground some necessary truths; and a grasp of (some of) these truths will indeed be part of what is involved in grasping the type. The trouble is that, in the example just reviewed and in the general run of cases, a grasp of the essential features of the type seems *to surpass* the information that one might plausibly regard as fully explicit in a standard explanation of the type; and yet it is not *deductively* unpacked from that information. Of course, that way of putting the point is not wholly happy, since in the rough and tumble of a normal education, there will in general be no uncontroversial way of fixing on what exactly that information has been. Still, in cases where a type is fixed a by picture or a diagram, as *cube*, or *quintet* may be, our capacities for the extraction of general propositional information remains—although, from an everyday perspective, a completely unremarkable part of ordinary intelligence—philosophically puzzling. Some necessities that go with the very notion of a type are epistemologically unproblematic, viz. those that are explicitly fixed by determining what counts as permissible variation in tokens of the type and what does not. The problem is that there will be a large range of generalisations, like those exploited in settling, in one of the various ways illustrated, that a cube has twelve edges, which we think a normally intelligent and receptive subject ought to 'get' but which were not made explicit. It takes *insight*, we might say, to grasp the essential characteristics of a type, and the insight is not in general that of foresight of the logical consequences of an explicit characterisation.

Now consider an excerpt from Part I of Wittgenstein, *Remarks on the Foundation of Mathematics* (MIT 1983)— I'll interpolate interpretative remarks:

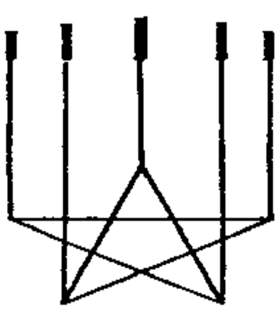
§25: But how about when I ascertain that this pattern of lines:

(a)

is like-numbered with this pattern of angles:

(b)

(I have made the patterns memorable on purpose) by correlating them:



(c)

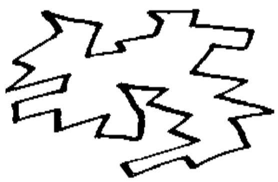
> Now what do I ascertain when I look at this figure? What I see is a star with threadlike appendages. —

— That is, the *empirical* content of what one sees is 'a star with threadlike appendages.' How do we elicit any *mathematical* (as Wittgenstein puts it below, *non-temporal*) content out of it? Well, it seems he wants to say, by conferring a certain kind of use on the picture:

> **§26** But I can make use of the figure like this: five people stand arranged in a pentagon; against the wall are wands, like the strokes in (a); I look at the figure (c) and say: "I can give each of the people a wand".
> I could regard figure (c) as a schematic *picture* of my giving five men a wand each.

He then contrasts regarding the figure as a 'schematic picture' in that kind of way with a purely empirical way of taking it:

> **§27** For if I first draw some arbitrary polygon:



> and then some arbitrary series of strokes
>
> | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
>
> I can find out by correlating them whether I have as many angles in the top figure as strokes in the bottom one. (I do not know how it would turn out.) And so I can also say that by drawing projection-lines I have ascertained that there are as many strokes at the top of figure (c) as the star beneath has points. (Temporally!) In this way of taking it the figure is not like a mathematical proof

— but is more like the case —

> when I divide a bag of apples among a group of people and find that each can have just *one* apple).
>
> I can however conceive figure (c) as a mathematical proof. Let us give names to the shapes of the patterns (a) and (b): let (a) be called a "hand", *H*, and (b) a "pentacle", *P*.

That is, we now introduce *H* and *P* as *types*. And then

> I have proved that *H* has as many strokes as *P* as angles. And this proposition is once more non-temporal.
>
> **§30** The proposition proved by (c) now serves as a new prescription for ascertaining numerical equality: if one set of objects has been arranged in the form of a hand and another as the angles of a pentacle, we say the two sets are equal in number.
>
> **§31** "But isn't that merely because we have already correlated *H* and *P* and seen that they are the same in number?" — Yes, but if they were so in one case, how do I know that they will be so again now? — "Why, because it is of the *essence* of *H* and *P* to be the same in number."

And the *essential* properties of the types, of course, are changeless.

> — But how can you have brought *that* out

—viz. that it is of the *essence* of *H* and *P* to be the same in number —

> by correlating them? (I thought the counting or correlation merely yielded the result that these two groups before me were – or were not – the same in number.)

How can the empirical correlation of the two inscribed token figures have brought out that *H* and *P*, introduced as types as above, *essentially* sustain the correlation?

> — "But now, if he has an *H* of things and a *P* of things, and he actually correlates them, it surely isn't *possible* for him to get any result but that they are the same in number. — And that it is not possible can surely seen from the proof." — But *isn't* it possible? If, e.g., he—as someone else might say—omits to draw one of the correlating lines. But I admit that in an enormous majority of cases he will always get the same result, and, if he did not get it, would think something had put him out. And if it were not like this the ground would be cut away from under the whole proof. For we decide to use the proof-picture instead of correlating the groups; we do *not* correlate them, but *instead* compare the groups with those of the proof (in which indeed two groups are correlated with one another).

So, Wittgenstein apparently wants to say, when his baby picture proof convinces someone of an essential connection between the two displayed types, or paradigms, there is no *impossibility*—of a breakdown of the connection—which it brings us to 'see'. Rather, we are empirically sure that the connection it illustrates will be confirmed in no end of normal token cases, and that where it is not, we will want to cry "foul" for reasons we can independently attest; and we are thereby moved to *decide* to use it in a certain way. We do not read off an essential connection between *H* and *P*. Rather we are moved to so co-ordinate the types concerned that it now comes to belong to the essence of both to sustain the connection:

> **§32** I might also say as a result of the proof: "From now on an *H* and a *P* are called 'the same in number'."
> Or: the proof doesn't *explore* the essence of the two figures, but it does express what I am going to count as belonging to the essence of the figures from now on. —I deposit what belongs to the essence among the paradigms of language.

And similarly, he continues, generalising, for proof at large:

> The mathematician creates *essences*.

In short, Wittgenstein appears to hold that essential relations between types in particular, and mathematical necessities in general, are never properly viewed as matters of recognition and discovery.

What in principle should determine which is the better view — our intuitive cognitivism or Wittgenstein's deflationism? Is this a stand-off?

Defl. (i) A certain kind of naturalism might incline one to the deflationary perspective, (although this does not seem to have been an explicit motive of Wittgenstein.)

Defl. (ii) Deflationism's best argument is the want of *any* kind of convincing model, even in outline, of how non-inferential a priori recognition of essential (necessary) truth is supposed to work. But

Cog. (iii) Why exactly do we need an explanation in the first place? Why shouldn't we rest content with the idea that a certain judgemental capacity of ours is purely cognitive even though no account is in prospect of how it accomplishes what we take it to do? To what extent should the postulation of a cognitive capacity be hostage to the possibility of an account of how it works?

What is a stake here is presumably a *sui generis* capacity: a capacity to recognise apparently a priori and non-inferentially new necessary commitments that are neither explicitly axiomatic nor definitional. 'Sui generis' means that we cannot assume that an explanation of its workings should proceed by way of an assimilation, or subsumption of the capacities in question under something more familiar and better understood— (as when the navigational abilities of honey bees are explained by the sensitivity of their visual systems to polarised light, or the computational abilities of mathematical prodigies are explained by the ascription to them of certain sub-personal recursive routines.) The explanation of a sui generis capacity cannot be a special case of the explanation of anything else we do. But what other kind of account might reasonably be expected?

These considerations tend towards the thought that, if we *do* have a capacity of non-inferential yet productive recognition a priori of necessary truth, it might not be reasonable to expect any explanation of its working in the first place.

Defl. (iv) But then, absent that, why believe in it? Even if not an explanation of its working, ought we not at least to demand some *evidence* that a genuinely cognitive capacity is at work?

Cog. (v) Ample evidence for that is surely provided by the near universal agreement in judgement that these routines provoke: all (normal) people agree about their probative force and what it is that they show.

Defl. (vi) That is a weak point. The sense of humour e.g is sufficiently widely shared to make stand-up comedy a practicable profession, and it is readily conceivable that it might be near enough universally shared, even though the kinds of process involved were no different to what they actually are. That would not be enough to transform it —if it is not already so—into a genuinely cognitive capacity. By contrast, the near universal agreement about the significance of e.g. simple arithmetical picture and process proofs might wane: young children might increasingly not 'get' them, even though competent with counting and other simple arithmetical routines. The mathematical aspect of such constructions might increasingly elude us, rather as the ability to find one or the other aspect of the Necker cube in the standard diagram sometimes goes missing for some people. If that happened, it would do nothing to suggest that the competence we actually have is not purely cognitive.

Defl. (vii) Bats have a sui generis capacity to track the positions and movement of objects in their vicinity by echolocation. But we know this only because we have *independent knowledge* of the things that bats are thereby sensitive to— so know that they are getting something right—and have presumably been able to verify what kind of disruption to their abilities is involved if they are prevented from making the relevant noises and receiving echoes. Scientists have further been able to construct a physiologically attested account of how echolocational sensitivities are realised in their sensory and neural systems, thereby providing a best explanation of how they are able to get the relevant matters right. With basic arithmetical and other necessities, in contrast, we don't so much as get to first base for a project of that kind. For we have no independent check on the matters that, on the cognitivist view, simple picture and process proofs put us in touch with: there is no body of essential fact, given to us independently, of which our responsiveness to basic a priori routines could then be verified as enabling us to track.

In sum: the significance of the case against cognitivism provided by the failure, assuming it does fail, of the explanatory project, is qualified by the consideration that it is not clear that it is reasonable to expect an explanation. But the pro-cognitivist significance of *this* point is qualified in turn by the further consideration that nor do we have any independent evidence that our basic a priori judgements, insofar as they are of counterfactually general claims, are getting anything right.