

Esercizi svolti sull'integrazione di funzioni razionali fratte

Caso 2.a

$$1) \int \frac{3x-4}{x^2-5x+6} dx = \int \frac{3x-4}{(x-2)(x-3)} dx = \int \left(\frac{-2}{x-2} + \frac{5}{x-3} \right) dx = -2 \ln|x-2| + 5 \ln|x-3| + c$$

I calcoli:

$$\frac{A}{x-2} + \frac{B}{x-3} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)} \text{ per ogni } x \Leftrightarrow \begin{cases} A+B=3 \\ -3A-2B=-4 \end{cases} \Leftrightarrow \begin{cases} A=-2 \\ B=5 \end{cases}$$

$$2) \int \frac{x^3-2x^2+1}{x^2-3x} dx = \int \left(x+1 + \frac{3x+1}{x^2-3x} \right) dx = \frac{1}{2}x^2 + x - \frac{1}{3} \ln|x| + \frac{10}{3} \ln|x-3| + c$$

I calcoli:

$$(x^3-2x^2+1) = (x^2-3x)(x+1) + 3x+1$$

$$\int \frac{x^3-2x^2+1}{x^2-3x} dx = \int \left(x+1 + \frac{3x+1}{x^2-3x} \right) dx$$

$$\frac{3x+1}{x^2-3x} = \frac{A}{x} + \frac{B}{x-3} = \frac{(A+B)x-3A}{x^2-3x} \text{ per ogni } x \Leftrightarrow \begin{cases} A=-\frac{1}{3} \\ B=\frac{10}{3} \end{cases}$$

3)

$$\int \frac{2x-3}{x^3-3x^2} dx = -\frac{1}{3} \int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \frac{1}{3} \int \frac{1}{x-3} dx = -\frac{1}{3} \ln|x| - \frac{1}{x} + \frac{1}{3} \ln|x-3| + c = \frac{1}{3} \ln \frac{|x-3|}{|x|} - \frac{1}{x} + c$$

I calcoli:

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} = \frac{Ax^2-3Ax+Bx-3B+Cx^2}{x^2(x-3)} = \frac{2x-3}{x^3-3x^2} \text{ per ogni } x \Leftrightarrow \begin{cases} A+C=0 \\ -3A+B=2 \\ -3B=-3 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{1}{3} \\ B=1 \\ C=\frac{1}{3} \end{cases}$$

Caso 2.b

$$1) \int \frac{3x-4}{x^2-6x+9} dx = \frac{3}{2} \int \frac{2x-6}{(x-3)^2} + 5 \int \frac{1}{(x-3)^2} dx = 3 \ln|x-3| + \frac{-5}{x-3} + c$$

2)

$$\begin{aligned} \int \frac{1}{x(x^2-1)^2} dx &= \int \left(\frac{1}{x} - \frac{1}{2} \frac{1}{x-1} - \frac{1}{4} \frac{1}{(x+1)^2} - \frac{1}{2} \frac{1}{x-1} + \frac{1}{4} \frac{1}{(x-1)^2} \right) dx = \\ &= \ln x - \frac{1}{4(x-1)} - \frac{1}{2} \ln|x-1| + \frac{1}{4(x+1)} - \frac{1}{2} \ln(x+1) + c = \\ &= \ln x - \frac{1}{2} \ln|x^2-1| - \frac{1}{2} \frac{1}{x^2-1} + c \end{aligned}$$

I calcoli:

$$\begin{aligned} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-1} + \frac{E}{(x-1)^2} &= \\ = \frac{-Ax^2 + Ax^4 + A - Bx^2 - Bx^3 + Bx^4 + Bx + Cx - 2Cx^2 + Cx^3 + Dx^3 + Dx^4 - Dx^2 - Dx + Ex^3 + 2Ex^2 + Ex}{x(x+1)^2(x-1)^2} \end{aligned}$$

$$\begin{cases} A+B+D=0 \\ -B+C+D+E=0 \\ -2A-B-2C-D+2E=0 \\ B+C-D+E=0 \\ A=1 \end{cases} \quad \text{La cui soluzione è: } \left\{ A=1, E=\frac{1}{4}, D=-\frac{1}{2}, C=-\frac{1}{4}, B=-\frac{1}{2} \right\}$$

Caso 2.c

$$1) \int \frac{1}{2x^2 + 1} dx = \int \frac{1}{(\sqrt{2}x)^2 + 1} dx = \frac{1}{2} \sqrt{2} \arctan x\sqrt{2} + c$$

$$2) \int \frac{x+1}{x^2+9} dx = \frac{1}{2} \int \frac{2x}{x^2+9} dx + \int \frac{1}{x^2+9} dx = \frac{1}{2} \ln(x^2+9) + \frac{1}{9} \int \frac{1}{\left(\frac{x}{3}\right)^2 + 1} dx = \\ = \frac{1}{2} \ln(x^2+9) + \frac{1}{3} \arctan \frac{x}{3} + c$$

$$3) \int \frac{1}{x^2+6x+10} dx = \int \frac{1}{(x-3)^2+1} dx + c = \arctan(x-3) + c$$

$$4) \int \frac{1}{x^3+x^2+x} dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+x+1} \right) dx = \ln x - \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int \frac{1}{x^2+x+1} dx = \\ = \ln x - \frac{1}{2} \ln(x^2+x+1) - \frac{\sqrt{3}}{3} \arctan \frac{1}{3} (2x+1)\sqrt{3} + c$$

I calcoli:

$$\frac{A}{x} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2 + Ax + A + Bx^2 + Cx}{x(x^2+x+1)} \text{ per ogni } x$$

$$\begin{cases} A+B=0 \\ A+C=0 \\ A=1 \end{cases} \text{ la cui soluzione è: } \{A=1, C=-1, B=-1\}$$

$$\frac{1}{2} \int \frac{1}{x^2+x+1} dx = \frac{1}{2} \int \frac{1}{\left(x+\frac{2}{3}\right)^2 + \frac{3}{4}} dx = \frac{1}{2} \frac{4}{3} \frac{\sqrt{3}}{2} \int \frac{\frac{2}{\sqrt{3}}}{\left(x+\frac{1}{2}\right)^2 + 1} dx = \frac{\sqrt{3}}{3} \arctan \frac{\sqrt{3}}{3} (2x+1) + c$$