MATEM. PER L'ECONOMIA LEZ. 9ª

LU 3/11/25

E.D.O. RICONDUCIBILE A VAR. SEPARABILI DEL I ORDINE:

A $\quad y' = g(ax+by)$

$ax+by$: COMBIN. LIN. DI x E y

PONGO $z = ax + by \implies$

$dz = a\,dx + b\,dy \implies$ DIVIDO PER dx

$$\frac{dz}{dx} = a + b\frac{dy}{dx}$$ CIOÈ $z' = a + by'$

PASSO 1: $a + b\,g(z) = 0 \implies g(z) = -\frac{a}{b}$ PUÒ ESSERE UNA SOLUZ.

PASSO 2: SE $g(z) \neq -\frac{a}{b} \implies$

$$\frac{dz}{dx} = a + b\,g(z) \implies$$

$$\frac{1}{a+b\,g(z)}\,dz = dx$$

ES. $y' = \frac{1}{x+y}$

PONGO $z = x+y \Rightarrow dz = dx + dy$

$\Rightarrow$ DIVIDO PER dx : $\frac{dz}{dx} = 1 + \frac{dy}{dx}$

CIOÈ $\frac{dz}{dx} = 1 + \frac{1}{z} = \frac{z+1}{z}$ VAR. SEPAR.

$$\frac{z}{z+1}\, dz = dx$$

PASSO 1 : $z+1=0$ È SOLUZ.

$\Rightarrow x+y = -1 \Rightarrow$

$\Rightarrow y = -x-1$

VERIFICO $y' = -1$ e $-1 = \frac{1}{-1}$

PASSO 2 : $\int \frac{z}{z+1}\, dz = \int dx$

$$\int \frac{z+1-1}{z+1}\, dz = x + c$$

$$z - \ln|z+1| = x + c$$

$$x+y-\ln|x+y+1| = x + c$$

$$\Longrightarrow y - \ln|x+y+1| = c \qquad (c \geq 0)$$

$$y - c = \ln|x+y+1|$$

$$e^{y-c} = |x+y+1|$$

$$e^{y}\cdot K = |x+y+1|$$

CIOÈ $x+y+1 = K e^{y}$ $K \in \mathbb{R}$

È LA SOLUZ. GEN. IN FORMA IMPLICITA y^*

B E.D.O. I ORDINE OMOGENEA

$$y' = g\left(\frac{y}{x}\right) \qquad x \neq 0$$

PONGO $z = \frac{y}{x} \Longrightarrow x \cdot z = y$

$\Longrightarrow$ DERIVO $dy = z\,dx + x\,dz$

$$\Longrightarrow \frac{dy}{dx} = \underbrace{z + x \cdot \frac{dz}{dx} = g(z)}$$

$$\Longrightarrow x\frac{dz}{dx} = g(z) - z \Longrightarrow$$

$$\frac{dz}{dx} = \frac{g(z)-z}{x}$$

PASSO 1: $g(z)-z=0 \Rightarrow g(z)=z$

PUÒ ESSERE UNA SOLUZ.

PASSO 2: SE $g(z) \neq z \Rightarrow$ VAR. SEP.

$$\int \frac{1}{g(z)-z}\,dz = \int \frac{1}{x}\,dx$$

ES. $y' = \frac{y}{x} + 1$

PONGO $z = \frac{y}{x} \Rightarrow x \cdot z = y \Rightarrow$

$dy = z\,dx + x\,dz$ DIVIDO PER dx

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

È y'

$$\cancel{z} + 1 = \cancel{z} + x\frac{dz}{dx}$$

$$\frac{1}{x} = \frac{dz}{dx} \Rightarrow$$

$$\int \frac{1}{x}\,dx = \int dz$$

$$z = \ln|x| + c$$

CIOÈ $\frac{y}{x} = \ln|x| + c$ $\quad x \neq 0$

$$y^* = x\ln|x| + cx \quad c \in \mathbb{R}$$

NEL PASSO 1 TROVIAMO UN INTEGRALE PARTICOLARE CON $c=0$

$$\bar{y} = x\ln|x|$$

$$\bar{y}' = 1 \cdot \ln|x| + x \cdot \frac{1}{x} = \ln|x| + 1$$

$$\bar{y}' = \frac{\bar{y}}{x} + 1 \quad \checkmark$$

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g) ES. 1 $\quad y' = e^{x+y}$ $\quad$ VAR. SEP.

$$y' = e^x \cdot e^y \implies \frac{dy}{dx} = e^x \cdot e^y$$

$$\implies \int \frac{1}{e^y}\, dy = \int e^x\, dx$$

$$-e^{-y} = e^x + c$$

$$e^{-y} = -e^x + c$$

$$e^{-y} = e^{\frac{1}{x}} + c$$

SOLUZ. GENER.

$$y^* = -\ln(-e^x + c) \quad c \in \mathbb{R}$$

EQUAZIONI DIFFERENZIALI LINEARI DEL 1° ORDINE A COEFF. COSTANTI

$$a y' + b y = 0 \qquad \text{OMOGENEA}$$

$$a y' + b y = g(x) \qquad \text{NON OMOG.}$$

PRIMA RISOLVO L'OMOGENEA, POI TROVO UNA SOLUZ. PARTICOLARE DELLA COMPLETA :

<u>PASSO 1</u> : SEMPRE $y(x) = 0$ RISOLVE E.D.L.O.

<u>PASSO 2</u> : $y' = -\frac{b}{a} y \qquad a \neq 0$

$\Longrightarrow$ SOLUZ. $y^*(x) = c e^{-\frac{b}{a}x}$

INFATTI $y^{*\prime} = c\, e^{-\frac{b}{a}x}\left(-\frac{b}{a}\right) =$

$= \left(-\frac{b}{a}\right) \cdot y^*$ OK ✓

ES. RISOLVI

$$5y' + 3y = 0$$

$$y' = -\frac{3}{5}y$$

$$y^*_{om.}(x) = c\,e^{-\frac{3}{5}x} \quad \text{SOL. OMOG.}$$

ES. RISOLVI

$$5y' + 3y = 4x + 1$$

PONGO $\bar{y} = ax + b$

$\bar{y}' = a$ SOSTITUISCO

$$5a + 3ax + 3b = 4x + 1$$

UGUAGLIANDO I MONOMI SIMILI

$$\begin{cases} 3a = 4 \\ 5a + 3b = 1 \end{cases} \longrightarrow \begin{cases} a = \frac{4}{3} \\ \frac{20}{3} + 3b = 1 \end{cases}$$

$$\begin{cases} a = \frac{4}{3} \\ 3b = 1 - \frac{20}{3} \end{cases} \qquad \begin{cases} a = \frac{4}{3} \\ b = -\frac{17}{9} \end{cases}$$

LA SOLUZ. COMPLETA È

$$y^*(x) = \underbrace{c\, e^{-\frac{3}{5}x}}_{y^*_{OM}} + \underbrace{\frac{4}{3}x - \frac{17}{9}}_{\bar{y}}$$

EQUAZ. DIFF. LINEARI DEL II ORDINE A COEFF. COSTANTI

$$y'' + a y' + b y = g(x)$$

$$y'' + a y' + b y = 0 \quad (a, b \in \mathbb{R}$$

OMOGENEA

SI ASSOCIA IL POLINOMIO CARATTERISTICO DI 2° GRADO

$$\lambda^2 + a\lambda + b = 0$$

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

1° CASO: $\Delta > 0 \quad \lambda_1, \lambda_2 \in \mathbb{R}$

SOL. $y^*_{OM}(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

2° CASO: $\Delta = 0 \quad \lambda_1 = \lambda_2 \in \mathbb{R}$

SOL. $y^*_{OM}(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

3° CASO : $\Delta < 0 \quad \lambda_1, \lambda_2 \in \mathbb{C}$

ES. $y'' - y = 0$

$$\lambda^2 - 1 = 0$$

$$\lambda_{1,2} = \pm 1$$

$$y^*_{OM}(x) = c_1 e^{x} + c_2 e^{-x}$$

MA LE SOL. POSSONO ESSERE

$$y^*_{OM}(x) = c_1 \cos(x) + c_2 \operatorname{sen}(x)$$

DELL' E.D.L. 2° $\quad y'' + y = 0$

$$\lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i \in \mathbb{C}$$

ES. $y'' - y = x^2 + 2x$

$$\bar{y} = \alpha x^2 + \beta x + \gamma$$

$$\bar{y}' = 2\alpha x + \beta$$

$$\bar{y}'' = 2\alpha$$

$$2\alpha - \underline{\underline{\alpha x^2}} - \beta x - \gamma = \underline{\underline{x^2}} + 2x$$

$$\begin{cases} -\alpha = 1 \\ -\beta = 2 \\ 2\alpha - \gamma = 0 \end{cases} \qquad \begin{cases} \alpha = -1 \\ \beta = -2 \\ +2 + \gamma = 0 \\ \gamma = -2 \end{cases}$$

$$y^*(x) = y^*_{OH}(x) + \bar{y}(x)$$

$$= c_1 e^x + c_2 e^{-x} - x^2 - 2x - 2$$

ES. 1 DISPENSA 1ª

$$\begin{cases} 4y' + 2y = 1 \\ y(-1) = 2 \end{cases}$$

1) $4y' + 2y = 0$

$2y' + y = 0$

$y' = -\frac{1}{2}y$

$y^*_{om}(x) = c\,e^{-\frac{1}{2}x}$

$\bar{y}(x) = K$

$\bar{y}' = 0$ $\quad 2K = 1$

$\bar{y}'' = 0$ $\quad K = \frac{1}{2}$

$y^*(x) = c\,e^{-\frac{1}{2}x} + \frac{1}{2}$

$y^*(-1) = c\,e^{\frac{1}{2}} + \frac{1}{2} = 2$

$$C e^{\frac{1}{2}} = \frac{3}{2}$$

$$C = \frac{3}{2} \frac{1}{\sqrt{e}} \in \mathbb{R}$$

ES.2 DISPENSA 1^a

$$\begin{cases} 2y' - y = -4 \\ y(1) = 1 \end{cases}$$

$$2y' - y = 0 \quad \text{OMOG.}$$

$$y' = \frac{1}{2} y$$

$$y^*_{om}(x) = C e^{\frac{1}{2}x}$$

$$\bar{y} = K \quad \bar{y}' = 0$$

$$-K = -4 \quad K = 4$$

$$y^*(x) = C e^{\frac{1}{2}x} + 4$$

$$y(1) = C e^{\frac{1}{2}} + 4 = 1$$

$$c = \frac{1-4}{e^{\frac{1}{2}}} = \frac{-3}{\sqrt{e}} = -3e^{-\frac{1}{2}}$$

ES. 4 DISP. 2°

$$\begin{cases} y'' - 10y = -10x^2 \\ y(0) = \frac{1}{5} \\ y'(0) = 0 \end{cases}$$

$$y'' - 10y = 0$$

$$\lambda^2 - 10 = 0$$

$$\lambda^2 = 10$$

$$\lambda_{1,2} = \pm\sqrt{10}$$

$$y^*_{om}(x) = c_1 e^{\sqrt{10}x} + c_2 e^{-\sqrt{10}x}$$

$$\bar{y}(x) = \alpha x^2 + \beta x + \gamma$$

$$\bar{y}' = 2\alpha x + \beta; \quad \bar{y}'' = 2\alpha$$

$$2\alpha - 10(\alpha x^2 + \beta x + \gamma) = -10x^2$$

$$\begin{cases} -10\alpha = -10 \\ -10\beta = 0 \\ 2\alpha - 10\gamma = 0 \end{cases}$$

$$\begin{cases} \alpha = 1 \\ \beta = 0 \\ 2 - 10\gamma = 0 \end{cases} \longrightarrow \gamma = \frac{2}{10} = \frac{1}{5}$$

$$y^*(x) = c_1 e^{\sqrt{10}\,x} + c_2 e^{-\sqrt{10}\,x} + x^2 + \frac{1}{5}$$

$$y(0) = c_1 + c_2 + \cancel{\frac{1}{5}} = \cancel{\frac{1}{5}}$$

$$y'(x) = c_1 \sqrt{10}\, e^{\sqrt{10}\,x} + c_2(-\sqrt{10})e^{-\sqrt{10}\,x} + 2x$$

$$y'(0) = \sqrt{10}\, c_1 - \sqrt{10}\, c_2 = 0$$

$$\begin{cases} c_1 + c_2 = 0 \\ \cancel{\sqrt{10}}\, c_1 - \cancel{\sqrt{10}}\, c_2 = 0 \end{cases}$$

$$2c_1 = 0$$

$$c_1 = 0$$
$$c_2 = 0$$

ES. 5) DISP. 2ª

$$\begin{cases} y'' + y' - 3y = e^x \\ y(0) = -1 \\ y'(0) = -1 \end{cases}$$